Applications

- translation, rotation, reflection, and sheering.
- Rigid body motions and robotic motion planning
- Computer graphics.
- Solutions to ordinary differential equations

Topics

- Linear functions (ALA 7.1)
  - Composition, inverses, isomorphisms
- Linear transformations (ALA 7.2)
  - A catalog
- Affine transformations (ALA 7.3)
- Isometries (ALA 7.3)
- Linear systems (ALA 7.4)
  - Linear Operators (ALA 7.1)
  - Superposition principle

Optional material

Linearity

A strategy that we have embraced so far has been one of turning algebraic questions into geometric ones. Our foundation for this strategy has been the vector space, which allows us to reason about a wide range of objects (vectors, polynomials, word histograms, and functions) as "arrows" that we can add, stretch, flip and rotate. Our canonical approach to transforming one vector into an other has been through matrix-vector multiplication: we start with a vector $\underline{x}$ and create a new vector via the mapping $\underline{x} \mapsto A\underline{x}$.

Our goal in this lecture is to give you a brief introduction to the theory of linear functions, of which the function $f(\underline{x}) = A\underline{x}$, is a special case. Linear functions are also known as linear maps, or when applied to function spaces, linear operators. These functions lie at the heart of robotics, computer graphics, quantum mechanics, and dynamical systems. We will see that by introducing just a little bit more abstraction, we can reason about all of these different settings using the same mathematical machinery.

Linear Functions

We start with the basic definition of a linear function which captures the fundamental idea of linearity: it does not matter if we sum two vectors and then transform them via a linear function, or apply the linear function to each vector individually, and then sum their transformations.

More formally, let V and W be real vector spaces. A function $L: V \to W$ mapping the domain V to the codomain W is called linear if it obeys two basic rules:

$$\underset{(L1)}{L(\underline{v}+\underline{w}) = L(\underline{v}) + L(\underline{w})} \quad \text{and} \quad \underset{(L2)}{L(c\underline{v}) = cL(\underline{v})}$$

for all $\underline{v}, \underline{w} \in V$ and $c \in \mathbb{R}$.

Before looking at some common examples, we make a few comments:

- Setting $c=0$ in rule (L2) tells us that a linear function always maps the zero element $\underline{0} \in V$ to the zero element $\underline{0} \in W$ (note these are different zeros elements as they live in different vector spaces!).
- A commonly used trick for verifying linearity is to combine (L1) and (L2) into the single rule

$$L(c\underline{v} + d\underline{w}) = cL(\underline{v}) + dL(\underline{w}) \text{ for all } \underline{v}, \underline{w} \in V,\ c, d \in \mathbb{R} \quad (L)$$

- We can extend rule (L) to any finite linear combination:

$$L(c_1\underline{v}_1 + \cdots + c_k\underline{v}_k) = c_1 L(\underline{v}_1) + \cdots + c_k L(\underline{v}_k) \qquad (LL)$$

for all $c_1, \dots, c_k \in \mathbb{R}$ and $\underline{v}_1, \dots, \underline{v}_k \in V$.

Finally a quick note on terminology: we will use linear function and linear map interchangeably when V and W are both finite dimensional, linear transformation when $V = W$, and linear operator when V and W are function spaces.

Example: Zero, Identity, and Scalar Multiplication Functions

- The zero function $O(\underline{v}) = \underline{0}$ which maps any $\underline{v} \in V$ to $\underline{0} \in W$ is easily checked to satisfy rule (L) (both sides are zero!).

- The identity function $I(\underline{v}) = \underline{v}$, which leaves any vector $\underline{v} \in V$ unchanged satisfies rule (L) because both $I(c\underline{v} + d\underline{w}) = c\underline{v} + d\underline{w}$ and $cI(\underline{v}) + dI(\underline{w}) = c\underline{v} + d\underline{w}$.

- The scalar multiplication function $M_a(\underline{v}) = a\underline{v}$ which scales an element $\underline{v} \in V$ by the scalar $a \in \mathbb{R}$ defines a linear function from V to itself, with $M_0(\underline{v}) = O(\underline{v})$ and $M_1(\underline{v}) = I(\underline{v})$, appearing as special cases.

NOTE: we made no assumptions about V and W in the above beyond them being vector spaces. They could be Euclidean spaces, function spaces, or even matrix spaces, and our statements would be equally valid!

Example: Matrix Multiplication

Let $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$. Then the function $L(\underline{v}) = A\underline{v}$ is a linear function since:

$$A(c\underline{v} + d\underline{w}) = cA\underline{v} + dA\underline{w} \quad \text{for all } \underline{v}, \underline{w} \in \mathbb{R}^n \text{ and } c, d \in \mathbb{R}$$

by the basic properties of matrix-vector multiplication.

In fact, matrix-vector multiplications are not only a familiar example of linear maps between Euclidean spaces, they are the only ones!

Theorem: Every linear function $L: \mathbb{R}^n \to \mathbb{R}^m$ is given by matrix-vector multiplication, $L(\underline{v}) = A\underline{v}$, for some $A \in \mathbb{R}^{m \times n}$.

Proof: The key idea is to apply the linear combination property (LL) to the expansion $\underline{v} = v_1 \underline{e}_1 + \cdots + v_n \underline{e}_n$ of $\underline{v}$ in the standard basis of $\mathbb{R}^n$:

$$L(\underline{v}) = L(v_1 \underline{e}_1 + \cdots + v_n \underline{e}_n) \underset{(LL)}{=} v_1 L(\underline{e}_1) + v_2 L(\underline{e}_2) + \cdots + v_n L(\underline{e}_n)$$

$$= \underbrace{\begin{bmatrix} L(\underline{e}_1) & L(\underline{e}_2) & \cdots & L(\underline{e}_n) \end{bmatrix}}_{A} \overset{\underline{v}}{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}$$

$$= A\underline{v}.$$

Thus we have shown that the way to find the matrix representation of a linear transformation is to evaluate it on the basis elements, and then stack them into a matrix: $A = [\,L(\underline{e}_1)\ \ L(\underline{e}_2)\ \cdots L(\underline{e}_n)\,]$.

WARNING: Pay attention to the order of m and n: when $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$, from $\mathbb{R}^n$ to $\mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, with m rows and n columns!

Example: 2D rotations

Let's consider the function $R_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that rotates a vector $\underline{v} \in \mathbb{R}^2$ counter clockwise by θ radians. To find its matrix representation, we look at the figure below and apply a little high school trigonometry (SOHCAHTOA anyone?):

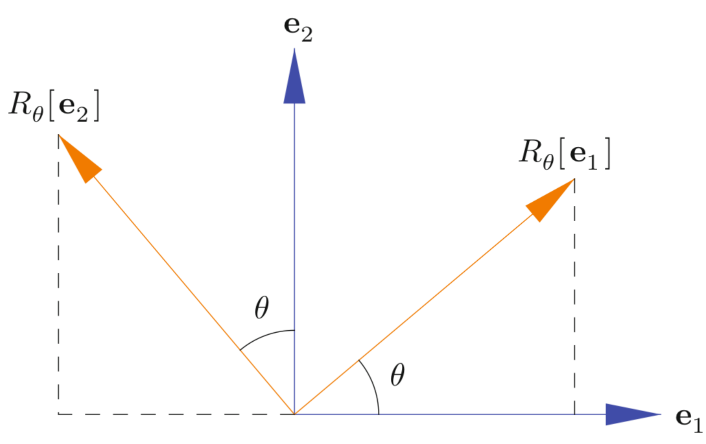


Figure 7.3. Rotation in $\mathbb{R}^2$.

Recalling that $\|\underline{e}_1\| = \|\underline{e}_2\| = 1$, and that rotating vectors preserves length, we have:

$$R_\theta(\underline{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad R_\theta(\underline{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix},$$

which, when stacked together, give the matrix representation $R_\theta(\underline{v}) = A_\theta \underline{v}$ with

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

This looks familiar! Indeed, this is the same expression we found when characterizing orthogonal 2×2 matrices. If we then apply $\underline{v} \mapsto A_\theta \underline{v}$ we obtain:

$$\hat{\underline{v}} = R_\theta(\underline{v}) = A_\theta \underline{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1\cos\theta - v_2\sin\theta \\ v_1\cos\theta + v_2\sin\theta \end{bmatrix}$$

which you can check are correct using trigonometry, but follow directly from the linearity of rotation.

Composition

Applying one linear transformation after another is called composition: let V, W, Z be vector spaces. If $L: V \to W$ and $M: W \to Z$ are linear functions, then the composite function $M \circ L: V \to Z$, defined by $(M \circ L)(\underline{v}) = M(L(\underline{v}))$ is also linear (easily checked to satisfy rule (L)).

This gives us a "dynamic" interpretation of matrix-matrix multiplication. If $L(\underline{v}) = A\underline{v}$ maps $\mathbb{R}^n$ to $\mathbb{R}^m$ and $M(\underline{w}) = B\underline{w}$ maps $\mathbb{R}^m$ to $\mathbb{R}^l$, so that $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{l \times m}$, then:

$$(M \circ L)(\underline{v}) = M(L(\underline{v})) = B(A\underline{v}) = (BA)\underline{v}$$

so that the matrix representation of $M \circ L: \mathbb{R}^n \to \mathbb{R}^l$ is the matrix product $BA \in \mathbb{R}^{l \times n}$. And, like matrix multiplication, composition of linear functions is in general not commutative (order of transformations matter!)

Example: composing rotations

Composing two rotations results in another: $R_\varphi \circ R_\theta = R_{\varphi+\theta}$, i.e., if we first rotate by θ, and then by φ, it is the same as rotating by $\theta + \varphi$. Using matrices:

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A_\varphi A_\theta = A_{\varphi+\theta} = \begin{bmatrix} \cos(\varphi+\theta) & -\sin(\varphi+\theta) \\ \sin(\varphi+\theta) & \cos(\varphi+\theta) \end{bmatrix}$$

Working out the LHS, we can deduce the well-known trigonometric addition formulas:

$$\cos(\varphi+\theta) = \cos\varphi\cos\theta - \sin\varphi\sin\theta, \quad \sin(\varphi+\theta) = \cos\varphi\sin\theta + \sin\varphi\cos\theta.$$

In fact, this counts as a proof!

Inverses

Just as with (square) matrices, we can define the inverse of a linear transformation. Let $L: V \to W$ be a linear function. If $M: W \to V$ is a linear function such that

$$L \circ M = I_W, \quad \text{and} \quad M \circ L = I_V,$$

where I_W and I_V are identity maps on W and V respectively, then M is the inverse of L and is denoted $M = L^{-1}$.

Example: Mapping polynomials $P^{(n)}$ to $\mathbb{R}^{n}$ and back again

Let $V = P^{(n)}$ be the space of polynomials of degree $\leq n$, and let $W = \mathbb{R}^{n+1}$.
Define the linear map $L: P^{(n)} \to \mathbb{R}^{n+1}$ as follows: for $p(x) = a_0 + a_1 x + \cdots + a_n x^n$,

$$L(p) = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix},$$

i.e., $L(p)$ stacks the coefficients of $p(x)$ into a vector $L(p) \in \mathbb{R}^{n+1}$.

The inverse map $L^{-1}(\underline{a})$ is simply the mapping that takes a vector $\underline{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$ and outputs the polynomial $L^{-1}(\underline{a})(x) = a_0 + a_1 x + \cdots + a_n x^n$. We check that it satisfies

$$L \circ L^{-1} = I_{\mathbb{R}^{n+1}} \quad \text{and} \quad L^{-1} \circ L = I_{P^{(n)}}$$

First $$(L \circ L^{-1})(\underline{a}) = L(L^{-1}(\underline{a})) = L(\underbrace{a_0 + a_1 x + \cdots + a_n x^n}_{L^{-1}(\underline{a})}) = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \underline{a}$$

for any $\underline{a} \in \mathbb{R}^{n+1}$, so that $L \circ L^{-1} = I_{\mathbb{R}^{n+1}}$. Next, we check, for any $p(x) = a_0 + a_1 x + \cdots + a_n x^n$:

$$(L^{-1} \circ L)(p) = L^{-1}(L(p)) = L^{-1}(\underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}}_{L(p)}) = L^{-1}(\underline{a}) = a_0 + a_1 x + \cdots + a_n x^n = p(x)$$

so that $L^{-1} \circ L = I_{P^{(n)}}$.

Because there exists an invertible linear map between $\mathbb{R}^{n+1}$ and $\underline{P}^{(n)}$ they are said to be *isomorphic*. As we saw earlier in the semester, this means that they "behave the same" and we can do vector space operations in either $\mathbb{R}^{n+1}$ or $\underline{P}^{(n)}$ whichever is convenient to us.

A more general statement can be made: any vector space of dimension n is isomorphic to $\mathbb{R}^n$, and so by studying Euclidean space, we in fact are gaining an understanding of *all finite dimensional vector spaces*.

Linear Transformations

Functions that map $\mathbb{R}^n \to \mathbb{R}^n$ are called linear transformations. They are special cases of the more general linear functions we saw above, but have a very nice geometric interpretation that help build intuition. In the tables below, we present some common transformations of $\mathbb{R}^2$, visualize their effect, and give their matrix representations.

TABLE 1 Reflections

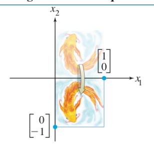
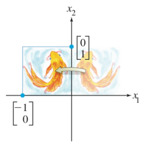
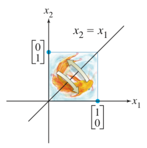
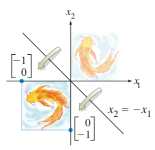
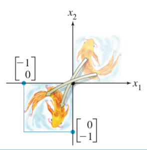
Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Reflection through the origin		$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

TABLE 2 Contractions and Expansions

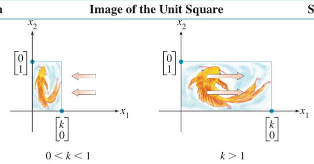
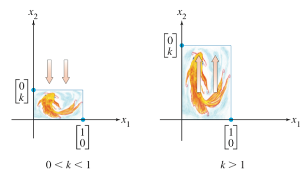
Transformation	Image of the Unit Square	Standard Matrix
Horizontal contraction and expansion		$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion		$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

TABLE 3 Shears

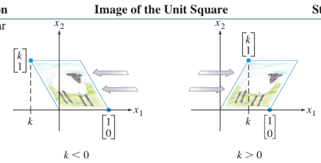
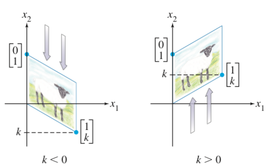
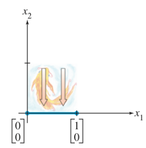
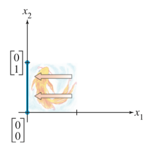
Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

TABLE 4 Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Affine Transformations

You'll notice that translations are conspicuously missing from the examples we've seen so far. That's because they are NOT linear functions! Rather, they are an example of slightly more general class of affine maps.

Specifically, we call a function $F: \mathbb{R}^n \to \mathbb{R}^m$ of the form

$$F(\underline{x}) = A\underline{x} + \underline{b},$$

where $A \in \mathbb{R}^{m \times n}$ and $\underline{b} \in \mathbb{R}^m$ an affine function. If $m = n$, then F defines an affine transformation.

For example, a translation that translates a vector $\underline{x}$ can be written as $F(\underline{x}) = \underline{x} + \underline{b}$, where $\underline{b}$ is the translation.

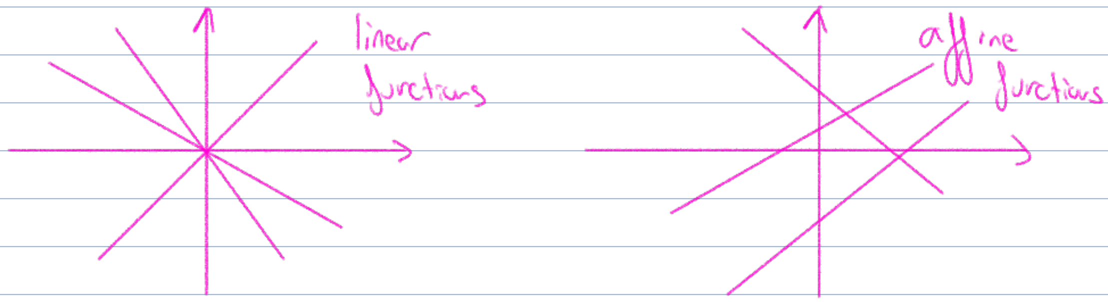
A perhaps more interesting example is the affine transformation

$$F(x,y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -y+1 \\ x-2 \end{bmatrix},$$

which has the effect of first rotating a vector 90° counter clockwise about the origin, and then translating the vector by $(1,-2)$.

INTUITION: You should think of linear functions as defining "lines through the origin" whereas affine functions define "lines with an offset"

On $\mathbb{R}^2$:



Isometry

A key property of rigid motions (translations, rotations, reflections), which are ubiquitous in robotics, mechanics, and computer graphics, is that they are distance preserving. In the context of Euclidean spaces, this means that applying the linear transformation $A\underline{x}$ to a vector $\underline{x}$ does not change its norm, i.e., $\|A\underline{x}\| = \|\underline{x}\|$.

We've argued informally that rotations and reflections are length preserving: let's now make this precise.

Theorem: A linear transformation $L(\underline{x}) = Q\underline{x}$ defines an isometry of $\mathbb{R}^n$ if and only if Q is an orthogonal matrix.

Proof: For $L(\underline{x}) = Q\underline{x}$ to be an isometry, we require $\|Q\underline{x}\| = \|\underline{x}\|$. But

$$\|Q\underline{x}\|^2 = \langle Q\underline{x}, Q\underline{x}\rangle = \underline{x}^T Q^T Q \underline{x} \quad \text{and} \quad \|\underline{x}\|^2 = \underline{x}^T\underline{x}$$

So that $\|Q\underline{x}\| = \|\underline{x}\|$ if and only if $\underline{x}^T QQ\underline{x} = \underline{x}^T\underline{x}$ for all $\underline{x} \in \mathbb{R}^n$. This is the case if and only if $Q^TQ = I$, which is the definition of an orthogonal matrix!

When characterizing affine isometries we need to work with distance between points, rather than length. Recall the distance function $\mathrm{dist}(\underline{x},\underline{y}) = \|\underline{x} - \underline{y}\|$. Then L is an affine isometry if $\mathrm{dist}(L(\underline{x}), L(\underline{y})) = \mathrm{dist}(\underline{x},\underline{y})$, i.e., if $\|L(\underline{x}) - L(\underline{y})\| = \|\underline{x} - \underline{y}\|$ for all $\underline{x},\underline{y} \in \mathbb{R}^n$.

To see that the translation $T(\underline{x}) = \underline{x} + \underline{b}$ satisfies this definition, note that

$$\text{dist}(T(\underline{x}), T(\underline{y})) = \|T(\underline{x}) - T(\underline{y})\| = \|\underline{x}+\underline{b} - (\underline{y}+\underline{b})\|$$
$$= \|\underline{x} - \underline{y}\| = \text{dist}(\underline{x}, \underline{y}).$$

ONLINE NOTES: let's include an ruiz or pybullet widget where we can move a camera and robotic arm around.

Linear Operators and Linear Systems (Optional Advanced Material)

Here we briefly highlight the generality of the machinery we've developed so far by dipping our toes into the world of linear operators. A linear operator is a linear transformation mapping between function spaces.

We'll look at one particular class of linear operators, called differential operators as they lie at the heart of differential equations, which we will be studying next.

We will work with the following function spaces:
- $C^0[0,1]$, the space of continuous functions defined on the interval $[0,1]$; and
- $C^1[0,1]$, the space of continuously differentiable functions over the interval $[0,1]$.

The derivative operator $D(f) = f'$ defines a linear operator $D: C^1[0,1] \to C^0[0,1]$. To see that this is the case, recall that

$$D(cf + dg) = (cf+dg)' = cf' + dg' = cD(f) + dD(g)$$

for any $f, g \in C^1[0,1]$ and $c, d \in \mathbb{R}$.

Just as with prior examples of linear maps, we can compose derivative operators to get higher order derivatives. For example

$$D \circ D(f) = D(D(f)) = D(f') = f''$$

is the 2nd order derivative, commonly denoted $D^2(f)$.

Another useful example of a linear operator is the evaluation operator, which evaluates a function f at a point x. For example, $E_0(f) = f(0)$ evaluates $f(x)$ at $x=0$. You should convince yourself that $E_x(f) = f(x)$ is a linear operator, by confirming that $E_x(cf+dg) = (cf+dg)(x) = cf(x) + dg(x) = cE_x(f) + dE_x(g)$ for any point x, functions f and g, and scalars $c, d \in \mathbb{R}$.

Linear Systems

Just as we could define linear systems of equations by writing $A\underline{x}=\underline{b}$, so too can we define general linear systems of the form

$$L(\underline{u})=\underline{f}$$

in which $L: \mathcal{U} \to V$ is a linear function between vector spaces, $\underline{f} \in V$ is an element of the codomain, while the solution $\underline{u} \in \mathcal{U}$ belongs to the domain.

We recover our familiar matrix-vector linear system $A\underline{u}=\underline{f}$ if $\mathcal{U}=\mathbb{R}^n$ and $V=\mathbb{R}^m$, and $L(\underline{u})=A\underline{u}$. However, we can express much more interesting problems in this framework.

Example: Consider a typical initial value problem

$$u''+u'-2u=f(t), \quad u(0)=1, \quad u'(0)=-1$$

for some unknown scalar function $u(t)$. First, we rewrite each equation in terms of derivative operators and evaluation operators

$$L_1(u)=u''+u'-2u=D^2(u)+D(u)-2(u)=(D^2+D-2)(u)=f(t)$$
$$L_2(u)=u(0)=E_0(u)=1$$
$$L_3(u)=u'(0)=E_0(D(u))=-1$$

If we then define the linear operator $M(u)$ and RHS $\underline{f}$ as

$$M(u)=\begin{bmatrix} L_1(u) \\ L_2(u) \\ L_3(u) \end{bmatrix} \quad \text{and} \quad \underline{f}=\begin{bmatrix} f(t) \\ 1 \\ -1 \end{bmatrix}$$

we can pose the initial value problem as a linear system $M(u)=\underline{f}$. In the above, what are the domain $\mathcal{U}$ and codomain V of the operator $M: \mathcal{U} \to V$?

The reason for introducing this extra layer of abstraction is that it lets us port over ideas from systems of linear equations. For example, the superposition principle holds here too!

We'll focus on solutions to homogeneous linear systems now, but if you're interested, §7.4 of ALA covers the general setting. The superposition principle here says that if a homogeneous linear system $L(\underline{z})=\underline{0}$, for $L: \mathcal{U} \to V$ a linear function, with two solutions $\underline{z}_1$ and $\underline{z}_2$ satisfying $L(\underline{z}_1)=\underline{0}$ and $L(\underline{z}_2)=0$, then any linear combination $c\underline{z}_1+d\underline{z}_2$ is also a solution. This follows immediately from the linearity of L:

$$L(c\underline{z}_1+d\underline{z}_2)=cL(\underline{z}_1)+dL(\underline{z}_2)=c\underline{0}+d\underline{0}=0.$$

In general, we have that if $\underline{z}_1, \dots, \underline{z}_k$ are all solutions to $L(\underline{z})=0$, then so is any linear combination $c_1\underline{z}_1 + \dots + c_k\underline{z}_k$. This means that the kernel

$$\ker L = \{ \underline{z} \in U \mid L(\underline{z}) = \underline{0} \} < U$$

forms a subspace of the domain space U.

<u>Example</u>: Consider the 2^{nd} order linear differential operator

$$L = D^2 - 2D - 3$$

which maps a function $u(t)$ to the function

$$L(u) = (D^2 - 2D - 3)(u) = D^2 u - 2Du - 3u$$
$$= u'' - 2u' - 3u.$$

The associated homogeneous system then encodes a homogeneous, linear, constant coefficient 2^{nd} order differential equation:

$$L(u) = u'' - 2u' - 3u = 0. \qquad \text{(ODE)}$$

Therefore, if we can characterize the kernel of L, we will have a general solution to this ODE.

Using techniques you would have seen in Math 1400, you can check that two linearly independent solutions (within the domain $C^2[0,1]$) to (ODE) are

$$u_1(t) = e^{3t} \quad \text{and} \quad u_2(t) = e^{-t}.$$

According to the superposition principle, every linear combination

$$u(t) = c_1 u_1(t) + c_2 u_2(t) = c_1 e^{3t} + c_2 e^{-t} \qquad (*)$$

is also a solution (try some values of c_1 and c_2 and check!). In fact, one can show that $\dim \ker L = 2$, and so any solution to (ODE) takes the form $(*)$.